Stability Analysis of a Plant Disease Model

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Abstract

We formulated a mathematical model with time delay, which represents the latent period of plant disease, and investigated how the delay affects the overall disease progression and, mathematically, how it affects the dynamics of the model. By analyzing the transcendental characteristic equation, we analyze the stability of the equilibria in terms of the delay. From the Nyquist criterion, an estimate on the length of delay is given for which the model which is stable in the absence of delay remains stable. Some explicit formulae, determining the stability and the direction of Hopf bifurcation periodic solutions, are obtained by use of the normal form theory. At last, numerical simulations are carried out to support the analytical results.

Keywords

Plant disease model, stability, normal form, Hopf bifurcation, periodic solution.

1. Introduction

Diseases caused by plant viruses in cassava, sweet potato and plantain are among the key constraints on sustainable production of these vegetatively propagated staple food crops [1]-[4]. Large investments are underway to develop new or more effective control strategies, including crop sanitation through removal of diseased plants and improved selection of planting materials, to alleviate poverty and malnutrition. A cultural control strategy considering the replanting or rouging infected plants is a widely accepted treatment for plant epidemics which involves periodic inspections followed by removal of the infected plants.

A simple model for plant disease with a continuous cultural control strategy, such as replanting and roguing or removing, is as follows

$$\begin{cases} \dfrac{dS(t)}{dt} = \sigma\phi - \beta S(t)I(t) - \eta S(t), \\ \dfrac{dI(t)}{dt} = \sigma(1-\phi) + \beta S(t)I(t) - (\eta+\omega)I(t), \end{cases} \tag{1.1}$$

where S, I are the number of the susceptible and the infected plants respectively, β is the transmission rate, σ is the continual replanting rate of the new plants, ϕ is the proportion for new susceptible plants, ω is the removal rate occurs for sanitation, η is the removal rate for death, and $\dfrac{1}{\eta}$ is either the harvest time or the end of reproductive lifetime of plants.

The extended forms of model (1.1) have been extensively studied, see [5]-[9] and the references therein. Meng and Li [7] extended model (1.1) by implementing continuous cultural control strategy and impulsive cultural control strategy and obtained

$$\begin{cases} \dfrac{dS(t)}{dt} = -\dfrac{\beta S(t)I(t)}{1+\alpha S(t)} - \mu S(t) + \omega I(t), & t \neq n\tau, \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} - (d+\omega)I(t), & t \neq n\tau, \\ S(t^+) = S(t) + \rho, & t = n\tau, \\ I(t^+) = (1-r)I(t), & t = n\tau, \end{cases} \tag{1.2}$$

where τ is a fixed positive constant and denotes the period of the impulsive effect, $n = 1, 2, \cdots$, and $0 \le \gamma \le 1$. The existence and stability of disease-free equilibrium and positive equilibrium are studied, and, from the Floquet's theorem and the small amplitude perturbation, sufficient conditions under which the infected plant free periodic solution is locally stable are obtain.

Model (1.2) ignores the infectives deriving from the infective cuttings and the reversion of the infected plants. For the crop planted from the cuttings taken form a previous one, the infected cuttings may be healthy due to reversion. van den Bosch *et al*. [5] extended model (1.1) to

$$\begin{cases} \dfrac{dS(t)}{dt} = \sigma\phi + \sigma(1-\phi)\dfrac{r(1-p)I(t)+S((t)}{(1-p)I(t)+S(t)} - \beta S(t)I(t) - \eta S(t), \\ \dfrac{dI(t)}{dt} = \sigma(1-\phi)\dfrac{(1-r)(1-p)I(t)}{(1-p)I(t)+S(t)} + \beta S(t)I(t) - (\eta+\omega)I(t), \end{cases} \tag{1.3}$$

where γ is the reversion probability of the cuttings from infected plants, the infected cuttings are visually or using diagnostic methods, discarded with probability p . They obtained a threshold parameter R_0 , determining the existence of a unique positive equilibrium, and found that the development of new and improved disease control methods for viral diseases of vegetatively propagated staple food crops ought to take evolutionary responses of the virus into consideration, and not doing so will lead to a risk of failure, which can result in considerable economic losses and increased poverty. In [5], van den Bosch *et al* did not study the qualitative properties of model (1.3) and only verified the stability of the positive equilibrium by numerical simulations.

Many plant diseases, such as *chlorotic leaf distortion* of sweet potato [10], *citrus black spot* [11], *colletotrichum gloeosporioides* [12], *alternaria alternata* [13] *et al*., possess latent period, *i.e.* the time elapsed between exposure to a pathogenic organism, and when symptoms and signs are first apparent. The earliest plant disease model is due to [14], where van der Plank used a delay differential equation to represent the density of host tissue first infected. Unfortunately, the delay differential equations are difficult to analyze [15, 16] and, despite widespread adoption of discrete time approximations to van der Plank's model in early simulations of plant disease [17, 18] and a number of often subtle mathematical analysis that followed [19]-[22], the model is now rarely used in theoretical studies [23]. However, it is so influential and still of significant historical interest. We use a nonnegative parameter τ to denote the latent period of the plant disease and formulate the delay differential model

$$\begin{cases} \frac{dS(t)}{dt} = \sigma\phi + \sigma(1-\phi)\frac{r(1-p)I(t)+S(t)}{(1-p)I(t)+S(t)} - \beta S(t-\tau)I(t-\tau) - \eta S(t), \\ \frac{dI(t)}{dt} = \sigma(1-\phi)\frac{(1-r)(1-p)I(t)}{(1-p)I(t)+S(t)} + \beta S(t-\tau)I(t-\tau) - (\eta+\omega)I(t). \end{cases} \quad (1.4)$$

For all $\tau \geq 0$, Zhang and Suo [24] proved that the disease-free equilibrium of model (1.4) is a critical equilibrium with a zero eigenvalue of (algebraic) multiplicity one when $R_0 = 1$. Then, they investigated the stability and the existence of steady-state bifurcation of model (1.4) near the non-hyperbolic disease-free equilibrium. In the present paper, we analytically investigated how the time delay affects the dynamical behaviors of model (1.4) near the positive equilibrium for all $\tau \geq 0$. The remainder of the paper is arranged as follows: Sections 2 discusses the existence and stability of the positive equilibrium, and the existence of Hopf bifurcation; Section 3 estimates the bound of time delay, and the positive equilibrium remains stable when the delay τ is smaller than it; Section 4 investigates the direction of the Hopf bifurcation; Section 5 performs numerical simulations to support the qualitative results and the last section makes the conclusions.

2. Existence and Stability of Equilibria

We adopt van den Bosch's assumptions on the parameters as $0 < p, \gamma, \phi < 1,\ \eta, \omega, \beta, \sigma > 0$.

We introduce

$$R_0 = \frac{(1-\phi)(1-r)(1-p)\eta^2 + \beta\sigma}{\eta(\eta+\omega)}.$$

Theorem 2.1. [24] *If $R_0 \leq 1$, (1.4) only has a disease-free equilibrium E_0 and if $R_0 > 1$, (1.4) has a unique positive equilibrium E^* except for the disease-free equilibrium E_0,*

where $S^* = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$, $I^* = \frac{\sigma(1-p)(1-r)(1-\phi) - S^*(\eta+\omega-\beta S^*)}{(\eta+\omega-\beta S^*)(1-p)}$

$E_0 = (\frac{\sigma}{\eta}, 0)$, $E^* = (S^*, I^*)$, $a_1 = \eta^2 p - \beta\sigma + \eta\omega p + \beta\sigma p + \omega^2 + \eta\omega$, $a_2 = -\beta(\eta p + \omega)$,

$a_0 = \sigma(p-1)(\phi(r-1) - r)(\eta+\omega)$.

Theorem 2.2. [24] *For $\tau = 0$. If $R_0 < 1$, the disease-free equilibrium E_0 is stable, while unstable if $R_0 \geq 1$.*

Theorem 2.3. [24] *For all $\tau > 0$. If $R_0 < 1$, the disease-free equilibrium E_0 is asymptotically stable; If $R_0 > 1$ and $\bar{\tau}_0 > \tau$, the disease-free equilibrium E_0 is unstable; If $R_0 = 1$ and $k_1 \neq 0$, the disease-free equilibrium E_0 is unstable; If $R_0 = 1$ and $k_1 = 0$, the disease-free equilibrium E0 is asymptotically stable, where*

$k_1 = \frac{\eta^3}{\sigma}(1-\phi)(1-r)(1-p)\big((p\eta+\omega) - \beta_0(\eta+\omega)\big)$, $\beta_0 = \frac{\eta}{\sigma}(((p-1)\eta\phi + (1-p)\eta)r$

$+(1-p)\eta\phi + \eta p + \omega)$, *and $\bar{\tau}_0$ is a positive number.*

To discuss the analytical properties of model (1.4) when $\tau \geq 0$, we first transfer it to the origin by $x_1 = S - S^*$, $x_2 = I - I^*$, and get

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t-\tau) + F(x(t), x(t-\tau)), \quad (2.1)$$

$$A=\begin{pmatrix} -h-\omega & \dfrac{S^*h}{I^*} \\ h & -\dfrac{S^*h}{I^*}-\eta-\omega \end{pmatrix},\quad B=\begin{pmatrix} -\beta I^* & -\beta S^* \\ \beta I^* & \beta S^* \end{pmatrix},$$

$$h=\frac{\sigma(\phi-1)(p-1)(r-1)I^*}{((p-1)I^*-S^*)^2},$$

$$F=\begin{pmatrix} \dfrac{\sigma(1-\phi)(p-1)(r-1)((p-1)x_2(t)-x_1(t))(S^*x_2(t)-I^*x_1(t))}{(I^*(p-1)-S^*)^2((p-1)(x_2(t)+I^*)-x_1(t)-S^*)}-\beta x_1(t-\tau)x_2(t-\tau) \\ -\dfrac{\sigma(1-\phi)(p-1)(r-1)((p-1)x_2(t)-x_1(t))(S^*x_2(t)-I^*x_1(t))}{(I^*(p-1)-S^*)^2((p-1)(x_2(t)+I^*)-x_1(t)-S^*)}+\beta x_1(t-\tau)x_2(t-\tau) \end{pmatrix}.$$

The characteristic equation of the linear part of Eq. (2.1) has the form

$$\lambda^2+p_1\lambda+p_2+e^{-\lambda\tau}(q_1\lambda+q_2)=0, \tag{2.2}$$

where

$$p_1=\omega+h+2\eta+\frac{hS^*}{I^*},\qquad p_2=\eta^2+(h+\eta)\omega+h\eta+\frac{h\eta S^*}{I^*},$$
$$q_1=\beta(I^*-S^*),\qquad q_2=\beta I^*(\omega+\eta)-\beta S^*\eta.$$

Theorem 2.4. *For $\tau=0$. If $p_1+q_1>0,\ p_2+q_2>0$ and $\Delta\geq 0$, the positive equilibrium E^* is a stable note; If $p_1+q_1>0,\ p_2+q_2>0$ and $\Delta<0$, the positive equilibrium E^* is a stable focus; If $p_2+q_2<0$, the positive equilibrium E^* is a saddle; If $p_1+q_1<0,\ p_2+q_2>0$ and $\Delta\geq 0$, the positive equilibrium E^* is an unstable note; If $p_1+q_1<0$ and $\Delta<0$, the positive equilibrium E^* is an unstable focus; If $p_1+q_1=0$ and $p_2+q_2=0$, the positive equilibrium E^* is a critical point possessing a pair of purely imaginary eigenvalues, and a Hopf bifurcation arises when the two eigenvalues cross through the imaginary axis, where $\Delta=(p_1+q_1)^2-4(p_2+q_2)$.*

In the following discussion, we suppose E^* is stable when $\tau=0$. It is can be verified that $p_2+q_2\neq 0$, which makes model (1.4) do not possess zero singularity of co-dimension one or more for all $\tau\geq 0$. Eq. (2.2) has a pair of purely imaginary roots $\pm iy, y>0$, if and only if the following equations hold

$$\begin{cases} q_2\cos(y\tau)+q_1y\sin(y\tau)=y^2-p_2, \\ q_2\sin(y\tau)-q_1y\cos(y\tau)=p_1y. \end{cases} \tag{2.3}$$

Adding up the squares of the two equations yields

$$y^4-(2p_2+q_1^2-p_1^2)y^2+p_2^2-q_2^2=0. \tag{2.4}$$

By the relationship between the roots and the coefficients of quadratic function, we have

Lemma 1. *If $p_2^2-q_2^2<0$ or $p_2^2-q_2^2=0,\ 2p_2+q_1^2-p_1^2>0$ Eq. (2.4) has a unique positive root; If $p_2^2-q_2^2>0,\ 2p_2+q_1^2-p_1^2>0$* and *$(2p_2+q_1^2-p_1^2)^2-4(p_2^2-q_2^2)\geq 0$, Eq. (2.4) has two positive roots (counting the multiplicity); If $(2p_2+q_1^2-p_1^2)^2-4(p_2^2-q_2^2)<0$* or *$(2p_2+q_1^2-p_1^2)^2-4(p_2^2-q_2^2)\geq 0,\ p_2^2-q_2^2\leq 0,\ 2p_2+q_1^2-p_1^2<0$, Eq. (2.4) has no root.*

The third case of Lemma 1 implies that the positive equilibrium E^* remains stable if $(2p_2+q_1^2-p_1^2)^2-4(p_2^2-q_2^2)<0$, or $2p_2+q_1^2-p_1^2<0$, $p_2^2-q_2^2\le 0$, $(2p_2+q_1^2-p_1^2)^2-4(p_2^2-q_2^2)\ge 0$.

Without losing generality, we assume that Eq. (2.4) has a unique positive root, and denote it as y_0. Eliminating sin($y_0\tau$) from Eq. (2.3), yields

$$\cos(y_0\tau)=\frac{q_2(y_0^2-p_2)-p_1q_1y_0^2}{q_2^2+q_1^2y_0^2}.$$

For the sake of convenience, we denote

$$\tau_j=\frac{1}{y_0}\arccos\{\frac{q_2(y_0^2-p_2)-p_1q_1y_0^2}{q_2^2+q_1^2y_0^2}\}+\frac{2j\pi}{y_0},\ j=0,1,2,\cdots. \qquad (2.5)$$

Theorem 2.5. *If* $2y_0^2-2p_2-q_1^2+p_1^2\neq 0$, *(1.4) undergoes Hopf bifurcations at the positive equilibrium* E^* *when* $\tau=\tau_j,\ j=0,1,2,\cdots$.

Proof. Differentiating Eq. (2.2) with respect to τ leads to

$$\left[2\lambda+p_1+(q_1-\tau(q_1\lambda+q_2))e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau}=\lambda e^{-\lambda\tau}(q_1\lambda+q_2),$$

which gives

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=-\frac{\lambda^2-p_2}{\lambda^2(\lambda^2+p_1\lambda+p_2)}-\frac{q_2}{\lambda^2(q_1\lambda+q_2)}-\frac{\tau}{\lambda}.$$

Let $\lambda=iy_0,\ \tau=\tau_j$. We have

$$\begin{aligned}\Re e\{(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=iy_0,\tau=\tau_j}\}&=-\frac{1}{y_0^2}\left(\frac{p_2^2-y_0^4}{(p_2-y_0^2)^2+y_0^2p_1^2}-\frac{q_2^2}{q_2^2+q_1^2y_0^2}\right)\\&=\frac{y_0^4+q_2^2-p_2^2}{y_0^2(q_2^2+q_1^2y_0^2)},\end{aligned} \qquad (2.6)$$

where $\Re e\{\}$ is the real part of $\cdot$.

Noticing that $q_2^2+p_2^2=y_0^4-y_0^2(2p_0+q_1^2-p_1^2)$. we have

$$\Re e\{(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=iy_0,\tau=\tau_j}\}=\frac{2y_0^2-2p_2-q_1^2+p_1^2}{q_2^2+q_1^2y_0^2}.$$

Then, **sgn** $\{\Re e\{(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=iy0,\tau=\tau_j}\}\}\neq 0$ if $2y_0^2-2p_2-q_1^2+p_1^2\neq 0$, where **sgn** is the signum function.

By use of the bifurcation theory [25]-[27], Hopf bifurcations undergo at $\tau=\tau_j,\ j=0,1,2,\cdots$. □

3. Estimation of the Delay to Preserve Stability

In the present section, we estimate the length of the delay for the positive equilibrium E^* keeps stable if τ is smaller that it.

Lemma 2. *(Nyquist criterion) If L is the arc length of a curve encircling the right half-plane, the curve* $\bar{P}_j(L)$ *will encircle the origin a number of times equal to the difference between the number of poles and the number of zeros* $\bar{P}_j(L)$ *in the right half-plane.*

We consider the linear part of Eq. (2.1) in $C([-\tau,\infty),\mathbb{R}^2)$ with the initial values

$$x_i(\xi)=\phi_i(\xi),\ i=1,2.$$

Taking Laplace transform on Eq. (2.1) leads to

$$X(s)-\Phi(0)=AX(s)+e^{-s\tau}B[X(s)+K(s)], \tag{3.1}$$

where $X(s)=(\overline{x}_1(s),\overline{x}_2(s))^T$, $\Phi(0)=(\phi_1(0),\phi_2(0))^T$, $K(s)=\int_{-\tau}^{0}e^{-st}x(t)\mathrm{d}t$ and $\overline{x}_1(s)$, $x_2(s)$ are the Laplace transforms of x_1, x_2 respectively.

Solving Eq. (3.1), we obtain

$$\overline{x}_2(s)=\frac{\Gamma(s,\tau)}{G(s,\tau)},$$

where

$$\Gamma(s,\tau)=(\phi_2(0)s+\phi_2(0)(\eta+h)+h\phi_1(0))+e^{-s\tau}\beta((k_1(s)I^*+k_2(s)S^*)s$$
$$+I^*(\phi_1(0)+\phi_2(0))+\eta(I^*k_1(s)+S^*k_2(s))),$$
$$G(s,\tau)=s^2+p_1s+p_2+e^{-s\tau}(q_1s+q_2),$$
$$k_i(s)=\int_{-\tau}^{0}e^{-s\tau}x_i(s)ds, i=1,2.n$$

Following along the lines of [28] and using Nyquist criterion, it can be shown that the conditions for local asymptotical stability of E^* are given by

$$\Im m\{G(iy_0,\tau)\}>0, \tag{3.2}$$

$$\Re e\{G(iy_0,\tau)\}=0, \tag{3.3}$$

where $\Im m(G(iy_0,\tau))$, is the imaginary part of $G(iy_0,\tau)$. Clearly, we have the following equations

$$p_1y_0>-q_1y_0\cos(y_0\tau)+q_2\sin(y_0\tau),$$
$$p_2-y_0^2=-q_2\cos(y_0\tau)-q_1y_0\sin(y_0\tau),$$

By use of Eq. (3.3), we have

$$y_0^2-|q_1|y0-|q_2|-|p_2|\ \le 0.$$

Let $y_+=\frac{|q_1|+\sqrt{q_1^2+4(|q_2|+|p_2|)}}{2}$. Then, we have $y_0\le y_+$.

Further, by virtue of Eqs. (3.2-3.3), we get

$$\begin{cases} y_0=\frac{p_2}{y_0}+\frac{q_2}{y_0}\cos(y_0\tau)+q_1\sin(y_0\tau),\\ p_1y_0>-q_1y_0\cos(y_0\tau)+q_2\sin(y_0\tau),\end{cases}$$

which leads to

$$-(q_1y_0+\frac{p_1q_2}{y_0})(\cos(y_0\tau)-1)+(q_2-p_1q_1)\sin(y_0\tau)$$
$$<\frac{p_1p_2}{y_0}+q_1y_0+\frac{p_1q_2}{y_0}. \tag{3.4}$$

Noticing that

$$1-\cos(y_0\tau)\le\frac{y_+^2\tau^2}{2},\quad \sin(y_0\tau)<y_+\tau,$$

we have

$$-(q_1y_0+\frac{p_1q_2}{y_0})(\cos(y_0\tau)-1)\le y_+^2\tau^2\sqrt{|p_1q_1q_2|}, \tag{3.5}$$

$$(q_2-p_1q_1)\sin(y_0\tau)\le|q_2-p_1q_1|y_+\tau \tag{3.6}$$

and

$$\frac{p_1p_2}{y_0}+q_1y_0+\frac{p_1q_2}{y_0}\le\sqrt{|p_1q_1|}(\sqrt{|p_2|}+\sqrt{|q_2|}). \tag{3.7}$$

On substituting Eq. (3.5- 3.7) into Eq. (3.4) yields

$$l_1\tau^2+l_2\tau\le l_3,$$

where $l_1=y_+^2\sqrt{|p_1q_1q_2|},\ l_2=|q_2-p_1q_1|y_+,\ l_3=\sqrt{|p_1q_1|}(\sqrt{|p_2|}+\sqrt{|q_2|})$.

Then, the positive equilibrium E^* is asymptotically stable if $\tau<\tau_+=\dfrac{-l_2+\sqrt{l_2^2+4l_1l_3}}{2l_1}$.

4. Direction of Hopf Bifurcation

It is interesting to determine the direction, stability and period of the periodical solutions bifurcated from the positive equilibrium E^*. In this section, following the idea of Hassard *et al*. [29], we investigate the properties of model (1.3) by use of the normal form and the center manifold theory [25]-[27].

First, rescale the time by $t\to t/\tau$ to normalize the delay of Eq. (2.1), and let $u_i(t)=x_i(\tau t),\ \tau=\tau_j+\mu,$ where μ is a small parameter. For convenience, we denote $u_i(t)$ as $x_i(t),\ i=1,2,$ respectively. Then, in $C=C([-1,\infty),\mathbb{R}^2)$, Eq. (2.1) can be rewritten as the following functional differential equation

$$\dot{x}(t)=L_\mu(x_t)+F(\mu,x_t), \tag{4.1}$$

where $x_t=x(t+_\theta)\in C$ and $L_\mu:C\to\mathbb{R},\ \ F:\mathbb{R}\times C\to\mathbb{R}$ are respectively defined by

$$L_\mu(\phi)=(\tau_j+\mu)(A\phi(0)+B\phi(-1)),$$

$$F(\mu,\phi)=(\tau_j+\mu)\begin{pmatrix}\dfrac{h(S^*\phi_2(0)-I^*\phi_1(0))(\phi_2(0)(p-1)-\phi_1(0))}{I^*((\phi_2(0)+I^*)(1-p)+\phi_1(0)+S^*)}-\beta\phi_1(-1)\phi_2(-1)\\ -\dfrac{h(S^*\phi_2(0)-I^*\phi_1(0))(\phi_2(0)(p-1)-\phi_1(0))}{I^*((\phi_2(0)+I^*)(1-p)+\phi_1(0)+S^*)}+\beta\phi_1(-1)\phi_2(-1)\end{pmatrix},$$

and $\phi(\theta)=(\phi_1(\theta),\phi_2(\theta))^T\in C.$

By the Riesz representation theorem, there exists a bounded variation matrix value function $\eta(\cdot,\cdot)$ such that

$$L_\mu\phi=\int_{-1}^0 d\eta(\theta,\mu)\phi(\theta),\phi\in C,$$

where $\theta\in[-1,0]$.

In fact, we can choose

$$\eta(\theta,\mu)=(\tau_j+\mu)(A\delta(\theta)+B\delta(\theta+1)).$$

Next, for $\phi\in C^1([-1,\infty),\mathbb{R}^2)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi}{d\theta}, & \theta = [-1,0]; \\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0. \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta = [-1,0]; \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then, Eq. (4.1) is equivalent to

$$x_t = A(\mu)x_t + R(\mu)x_t, \tag{4.3}$$

where $x_t = x(t+\theta)$.

For $\psi \in C^* = C^1([0,-1),\mathbb{R}^{2*})$, we define

$$A^*(0)\psi = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s = [-1,0]; \\ \displaystyle\int_{-1}^{0} \psi(-t)d\eta(t,0), & s = 0. \end{cases}$$

where $\mathbb{R}^{2*}$ is the space of all row 2-vectors.

Define the bilinear form between C and C^*

$$< \psi(s),\phi(\theta) >= \bar{\psi}(0),\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\eta)d\eta(\theta)\phi(\xi)d\xi, \tag{4.4}$$

where $\eta(\theta) = \eta(\theta,0)$, and $a \cdot b$ means $\sum_{i=1}^{n} a_i b_i$.

For the sake of computation, we rewrite $A(0),\ A^*(0)$ as $A,\ A^*$ respectively. Then, the operators A and A^* are adjoint, and $\pm iy_0\tau_j$ are the eigenvalues of them. We first need to compute the eigenvectors of A and A^* corresponding to $iy_0\tau_j$ and $-iy_0\tau_j$, respectively.

Let $q(\theta) = (1,\alpha)^T e^{iy_0\tau_j\theta}$ be the eigenvector of A corresponding to $iy_0\tau_j$. It follows from the definition of A that

$$\left(iy_0 I_{2\times 2} - (A + Be^{-iy_0\tau_j})\right)\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = 0,$$

where $I_{2\times 2}$ is the unit matrix of 2×2. Then, we have $\alpha = \dfrac{iy_0\beta S^* e^{-iy_0\tau_j} + \eta + h}{\dfrac{S^* h}{I^*} - \beta S^* e^{-iy_0\tau_j}}$.

Similarly, let $q^*(s) = D(1,\alpha^*)e^{-iy_0\tau_j s}$ be the eigenvector of A^* corresponding to $-iy_0\tau_j$, where D is a constant will be determined in the following. Then, we have

$$(1,\alpha^*)(iy_0 I_{2\times 2} + A + Be^{-iy_0\tau_j}) = 0,$$

and further have

$$\alpha^* = -\frac{iy_0 + \eta + h + \beta S^* e^{-iy_0\tau_j}}{h + \beta I^* e^{-iy_0\tau_j}}.$$

In order to guarantee $< q^*(s), q(\theta) >= 1$, which leads to

$$D = \frac{1}{1+\bar{\alpha}\alpha^* + \bar{K}},$$

where

$$K = \frac{-\beta I^* e^{iy_0\tau_j}}{(h+\beta I^* e^{iy_0\tau_j})(\beta S^* I^* e^{-iy_0\tau_j} - S^* h)}(\beta S^* I^* e^{-iy_0\tau_j} - \beta(S^*)^2 e^{-iy_0\tau_j}$$
$$-2hS^* - S^*\eta - iy_0 S^*)(h\beta S^* e^{-iy_0\tau_j} + \beta I^* e^{iy_0\tau_j} - iy_0 + \eta + h).n$$

In the remainder of this section, using the same notations as in [28], we compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Eq. (4.3) at $\mu = 0$, and define

$$z(t) = < q^*, x_t >, W(t,\theta) = x_t(\theta) - 2\Re\{z(t)q\}.$$

On the manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{4.5}$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{4.6}$$

z and $\bar{z}$ are local coordinates for C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is, we shall deal with the real solutions only. Since $\mu = 0$, we have

$$\dot{z}(t) = iy_0\tau_j z(t) + \overline{q^*} F(0, W(z,\bar{z},\theta) + 2\Re\{zq\})$$
$$:= iy_0\tau_j + \overline{q^*}(0) F_0(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{30}\frac{z^3}{6} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.$$

It follows from Eqs. (4.5-4.6) that

$$\begin{aligned} x_t(\theta) &= W(t,\theta) + 2\Re\{zq\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1,\alpha)^T e^{iy_0\tau_j\theta} z + (1,\bar{\alpha})^T e^{-iy_0\tau_j\theta}\bar{z} + \cdots \\ &= \begin{pmatrix} W_{20}^1(\theta)\frac{z^2}{2} + W_{11}^1(\theta)z\bar{z} + W_{02}^1(\theta)\frac{\bar{z}^2}{2} + e^{iy_0\tau_j\theta} z + e^{-iy_0\tau_j\theta}\bar{z} + \cdots \\ W_{20}^2(\theta)\frac{z^2}{2} + W_{11}^2(\theta)z\bar{z} + W_{02}^2(\theta)\frac{\bar{z}^2}{2} + \alpha e^{iy_0\tau_j\theta} z + \bar{\alpha} e^{-iy_0\tau_j\theta}\bar{z} + \cdots \end{pmatrix} \end{aligned}$$

By virtue of Eq. (4.2), we get

$$\begin{aligned} g(z,\bar{z}) &= \overline{q^*}(0) F(0, W(z,\bar{z},\theta) + \Re\{zq\}) \\ &= \tau_j \bar{D}(1, \overline{\alpha^*}) \begin{pmatrix} \dfrac{h(S^* x_{2t}(0) - I^* x_{1t}(0))(x_{2t}(0)(p-1) - x_{1t}(0))}{I^*((x_{2t}(0) + I^*)(1-p) + x_{1t}(0) + S^*)} - \beta x_{1t}(-1) x_{2t}(-1) \\ -\dfrac{h(S^* x_{2t}(0) - I^* x_{1t}(0))(x_{2t}(0)(p-1) - x_{1t}(0))}{I^*((x_{2t}(0) + I^*)(1-p) + x_{1t}(0) + S^*)} + \beta x_{1t}(-1) x_{2t}(-1) \end{pmatrix} \end{aligned}$$

$$=\tau_j\bar{D}(1-\overline{\alpha^*})\Big\{\frac{2h}{I^*(1-p)+S^*}\Big[W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\bar{z}+W_{02}^1(0)\frac{\bar{z}^2}{2}+z+\bar{z}\Big]^2$$

$$-\frac{h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}\Big[W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\bar{z}+W_{02}^1(0)\frac{\bar{z}^2}{2}+z+\bar{z}\Big]$$

$$\cdot[W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\bar{z}+W_{02}^2(0)\frac{\bar{z}^2}{2}+\alpha z+\bar{\alpha}\bar{z}]$$

$$-\frac{2hS^*(1-p)}{I^*(S^*+(1-p)I^*)}\Big[W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\bar{z}+W_{02}^2(0)\frac{\bar{z}^2}{2}+\alpha z+\bar{\alpha}\bar{z}\Big]^2$$

$$-\beta\Big[W_{20}^1(-1)\frac{z^2}{2}+W_{11}^1(-1)z\bar{z}+W_{02}^1(-1)\frac{\bar{z}^2}{2}+ze^{-iy_0\tau_j}+\bar{z}e^{iy_0\tau_j}\Big]$$

$$\cdot\Big[W_{20}^2(-1)\frac{z^2}{2}+W_{11}^2(-1)z\bar{z}+W_{02}^2(-1)\frac{\bar{z}^2}{2}+\alpha ze^{-iy_0\tau_j}+\bar{\alpha}\bar{z}e^{iy_0\tau_j}\Big]$$

$$+\frac{2h(S^*+2I^*(p-1))}{(S^*+(1-p)I^*)^2I^*}\Big[W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\bar{z}+W_{02}^1(0)\frac{\bar{z}^2}{2}+z+\bar{z}\Big]^2$$

$$\cdot[W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\bar{z}+W_{02}^2(0)\frac{\bar{z}^2}{2}+z+\bar{z}]\Big\}+\cdots.$$

Balancing above equation yields

$$g_{20}=\tau_j\bar{D}(1-\overline{\alpha^*})\Big[\frac{2h}{I^*(1-p)+S^*}-\frac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)}-\frac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)}-\beta\alpha e^{-2iy_0\tau_j}\Big],$$

$$g_{11}=\tau_j\bar{D}(1-\overline{\alpha^*})\Big[\frac{4h}{I^*(1-p)+S^*}-\frac{h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}(\bar{\alpha}+\alpha)-\frac{4hS^*(1-p)}{I^*(S^*+(1-p)I^*)}\alpha\bar{\alpha}-\beta(\alpha+\bar{\alpha})\Big],$$

$$g_{02}=\tau_j\bar{D}(1-\overline{\alpha^*})\Big[\frac{2h}{I^*(1-p)+S^*}-\frac{h(S^*+I^*(p-1))\bar{\alpha}}{I^*(S^*+(1-p)I^*)}-\frac{2hS^*(1-p)\bar{\alpha}^2}{I^*(S^*+(1-p)I^*)}-\beta e^{2iy_0\tau_j}\bar{\alpha}\bar{z}\Big],$$

$$g_{21}=\tau_j\bar{D}(1-\overline{\alpha^*})\Big[\frac{4h}{I^*(1-p)+S^*}(W_{11}^1(0)+\frac{1}{2}W_{20}^1(0))-\frac{h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}(\alpha W_{11}^1(0)n+W_{11}^2(0)$$

$$+\frac{\bar{\alpha}}{2}W_{20}^1(0)+\frac{\alpha}{2}W_{20}^2(0))-\frac{4hS^*(1-p)}{I^*(S^*+(1-p)I^*)}(\frac{\bar{\alpha}}{2}W_{20}^2(0)+\alpha W_{11}^2(0))$$

$$-\beta(\frac{\bar{\alpha}}{2}e^{iy_0\tau_j}W_{20}^1(-1)+\alpha W_{11}^1(-1)e^{-iy_0\tau_j}+\frac{1}{2}W_{20}^2(-1)e^{iy_0\tau_j}+W_{11}^2(-1)e^{-iy_0\tau_j})\Big].$$

Obviously, we obtain

$$\dot{W}=\dot{x}_t-\dot{z}q-\dot{\bar{z}}\bar{q}=\begin{cases}AW-2\Re\mathfrak{e}\{\overline{q^*}(0)F_0q(\theta)\}, & \theta\in[-1,0);\\ AW-2\Re\mathfrak{e}\{\overline{q^*}(0)F_0q(\theta)\}+F_0, & \theta=0.\end{cases} \tag{4.7}$$

which can be rewritten as

$$\dot{W}=AW+H(z,\bar{z},\theta), \tag{4.8}$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \tag{4.9}$$

On the center manifold C_0, we have

$$\begin{aligned}\dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \\ &= (W_{20}(\theta)z + W_{11}(\theta)\bar{z} + W_{30}(\theta)\frac{z^2}{3} + \cdots)(iy_0\tau_j z + g(z,\bar{z})) \\ &\quad + (W_{11}(\theta)z + W_{02}(\theta)\bar{z} + \cdots)(-iy_0\tau_j\bar{z} + \bar{g}(z,\bar{z})).\end{aligned}$$

By means of Eq. (4.7), it can be obtained that

$$(2iy_0\tau_j - A)W_{20}(\theta) = -H_{20}(\theta), \quad -AW_{11}(\theta) = -H_{11}(\theta), \cdots, \tag{4.10}$$

and further obtained that

$$\begin{aligned}H(z,\bar{z},\theta) &= -2\Re\mathfrak{e}\{\overline{q^*}F_0q(\theta)\} \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots)q(\theta) \\ &\quad -(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \bar{g}_{21}\frac{z\bar{z}^2}{2} + \cdots)\bar{q}(\theta).\end{aligned}$$

Balancing the coefficients of above equation yields

$$\begin{aligned}H_{20}(\theta) &= -(g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)), \\ H_{11}(\theta) &= -(g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta)), \\ H_{20}(\theta) &= -(g_{02}q(\theta) + \bar{g}_{20}\bar{q}(\theta)).\end{aligned}$$

By Eq. (4.8), the following equation is obtained

$$\dot{W}_{20}(\theta) = 2iy_0\tau_jW_{20}(\theta) - H_{20}(\theta) = 2iy_0\tau_jW_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{20}\bar{q}(\theta). \tag{4.11}$$

Then, we have

$$W_{20}(\theta) = \frac{g_{20}q(0)i}{y_0\tau_j}e^{iy_0\tau_j\theta} + \frac{\bar{g}_{02}\bar{q}(0)i}{3y_0\tau_j}e^{-iy_0\tau_j\theta} + e^{2iy_0\tau_j\theta}E_1, \tag{4.12}$$

where $E_1 \in \mathbb{R}^2$ is a constant vector.

By the second equation of Eqs. (4.10), we derive

$$\dot{W}_{11}(\theta) = g_{11}q(0)e^{iy_0\tau_j\theta} + \bar{g}_{11}\bar{q}(0)e^{-iy_0\tau_j\theta}.$$

Then, we get

$$W_{11}(\theta) = -\frac{ig_{11}q(0)}{y_0\tau_j}e^{iy_0\tau_j\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{y_0\tau_j}e^{-iy_0\tau_j\theta} + E_2, \tag{4.13}$$

where $E_2 \in \mathbb{R}^2$ is a constant vector.

In the sequel, we shall determine the vectors E_1 and E_2. By Eq. (4.8), we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2iy_0\tau_jW_{20}(0) - H_{20}(0)$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \tag{4.14}$$

By Eqs. (4.8-4.9), we get

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0)$$
$$+\tau_j\begin{pmatrix} \dfrac{2h}{I^*(1-p)+S^*} - \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} - \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} - \beta\alpha e^{-2iy_0\tau_j} \\ -\dfrac{2h}{I^*(1-p)+S^*} + \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} + \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} + \beta\alpha e^{-2iy_0\tau_j} \end{pmatrix}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_j.$$
$$\begin{pmatrix} \dfrac{4h}{I^*(1-p)+S^*} - \dfrac{2h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}\Re\mathfrak{e}\{\alpha\} - \dfrac{4hS^*(1-p)}{I^*(S^*+I^*(1-p))}\alpha^2 - 2\beta\Re\mathfrak{e}\{\alpha\} \\ \dfrac{-4h}{I^*(1-p)+S^*} + \dfrac{2h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}\Re\mathfrak{e}\{\alpha\} + \dfrac{4hS^*(1-p)}{I^*(S^*+I^*(1-p))}\alpha^2 + 2\beta\Re\mathfrak{e}\{\alpha\} \end{pmatrix}. \tag{4.15}$$

Note that

$$\left(i\tau_j y_0 I_{2\times2} - \int_{-1}^{0} e^{iy_0\tau_j\theta}d\eta(\theta)\right)q(0) = 0$$

and

$$\left(-iy_0\tau_j I_{2\times2} - \int_{-1}^{0} e^{-iy_0\tau_j\theta}d\eta(\theta)\right)\bar{q}(0) = 0,$$

we arrive at

$$\left(2i\tau_j I_{2\times2} - \int_{-1}^{0} e^{iy_0\tau_j\theta}d\eta(\theta)\right)E_1 =$$
$$\tau_j\begin{pmatrix} \dfrac{2h}{I^*(1-p)+S^*} - \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} - \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} - \beta\alpha e^{-2iy_0\tau_j} \\ -\dfrac{2h}{I^*(1-p)+S^*} + \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} + \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} + \beta\alpha e^{-2iy_0\tau_j} \end{pmatrix}.$$

By the definition of $\eta(\theta)$, we derive

$$E_1 = \begin{pmatrix} 2iy_0 + \eta + h + \beta I^* e^{-2iy_0\tau_j} & -\dfrac{S^*h}{I^*} + \beta S^* e^{-2iy_0\tau_j} \\ -h - \beta I^* e^{-2iy_0\tau_j} & 2iy_0 + \dfrac{S^*h}{I^*} + \eta + \omega - \beta S^* e^{-2iy_0\tau_j} \end{pmatrix}^{-1} \times$$
$$\begin{pmatrix} \dfrac{2h}{I^*(1-p)+S^*} - \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} - \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} - \beta\alpha e^{-2iy_0\tau_j} \\ -\dfrac{2h}{I^*(1-p)+S^*} + \dfrac{h(S^*+I^*(p-1))\alpha}{I^*(S^*+(1-p)I^*)} + \dfrac{2hS^*(1-p)\alpha^2}{I^*(S^*+(1-p)I^*)} + \beta\alpha e^{-2iy_0\tau_j} \end{pmatrix}. \tag{4.16}$$

Similarly, substituting Eqs. (4.13-4.15) into Eq. (4.14) leads to

$$E_2 = -\begin{pmatrix} -\beta I^* - \eta - h & \dfrac{S^*h}{I^*} - \beta S^* \\ h + \beta I^* & \beta S^* - \dfrac{S^*h}{I^*} - \eta - \omega \end{pmatrix}^{-1} \times$$

$$\begin{pmatrix} \frac{4h}{I^*(1-p)+S^*} - \frac{2h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}\Re\{\alpha\} - \frac{4hS^*(1-p)}{I^*(S^*+I^*(1-p))}\alpha^2 - 2\beta\Re\{\alpha\} \\ -\frac{4h}{I^*(1-p)+S^*} + \frac{2h(S^*+I^*(p-1))}{I^*(S^*+(1-p)I^*)}\Re\{\alpha\} + \frac{4hS^*(1-p)}{I^*(S^*+I^*(1-p))}\alpha^2 + 2\beta\Re\{\alpha\} \end{pmatrix}. \quad (4.17)$$

From Eqs. (4.12-4.13,4.16-4.17), we can calculate g_{21} and derive the following values

$$c_1(0) = \frac{i}{2y_0\tau_j}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \qquad \kappa_2 = -\frac{\Re\{c_1(0)\}}{\Re\{\lambda'(\tau_j)\}},$$
$$b_2 = \Re\{c_1(0)\}, \qquad T_2 = -\frac{\Im\{c_1(0)\} + \kappa_2\Im\{\lambda'(\tau_j)\}}{\tau_j y_0}. \quad (4.18)$$

These formulas give a description of the Hopf bifurcation periodic solutions of Eq. (4.1) at $\tau = \tau_j$ on the center manifold. On the basis of the above discussion, we have the following results.

Theorem 4.1. *In Eq. (4.18), the sign of* $_2$ *determines the directions of Hopf bifurcation: if* $k_2 > 0$ *(or* $k_2 < 0$*), the Hopf bifurcation is supercritical (or subcritical)and the bifurcating periodical solutions exist for* $\tau > \tau_j$ *(or* $\tau < \tau_j$*);* b_2 *determines the stability of the periodical solution: the bifurcating solutions are stable (or unstable) if* $b_2 < 0$ *(or* $b_2 > 0$*) and* T_2 *determines the period of the bifurcated periodical solutions: the period increases (or decreases) if* $T_2 > 0$ *(or* $T_2 < 0$*).*

5 Numerical Simulations

In this section, by using the "dde23" package in Matlab, we present some numerical results associated with model (1.4) for some particular values of the parameters given in Table. 1. For ecological justification behind the choice of numerical values and related information, interested readers may consult the literature [5] and the references. Choosing $\phi = 0.07$, $p = 0.0064$, we obtain $R_0 \approx 21.60880362$ .Then, model(1.4) has a unique positive equilibrium $S^* \approx 0.05223044016$ $I^* \approx 6.476321355$ .It follows form the formulas in Theorem 4.1 that $\tau_0 = 38.21688903$, $b_2 = -0.0002645313913$ $k_2 = 0.5430753270$, and $T_2 = -0.005158896720$, which means Hopf bifurcations occur at $\tau < \tau_0$ when τ is through the critical values τ_0 . Fig. 1 exhibits the positive equilibrium is stable for $\tau = 35$, and Fig. 2 shows that Hopf bifurcation occurs at the positive equilibrium for $\tau = 38.22$.

Table 1. Model parameters, their interpretation and values.

Parameter	Description	Value
η	harvest rate	0.002
σ	planting rate	0.015
ϕ	fraction planted from *in vitro* propageted, virus free, material	(0, 1)
p	the probability of detecting an infected cuttings	(0, 1)
γ	reversion ratio	0.08
ω	roguing rate	0.0003
β	transmission rate	0.0064

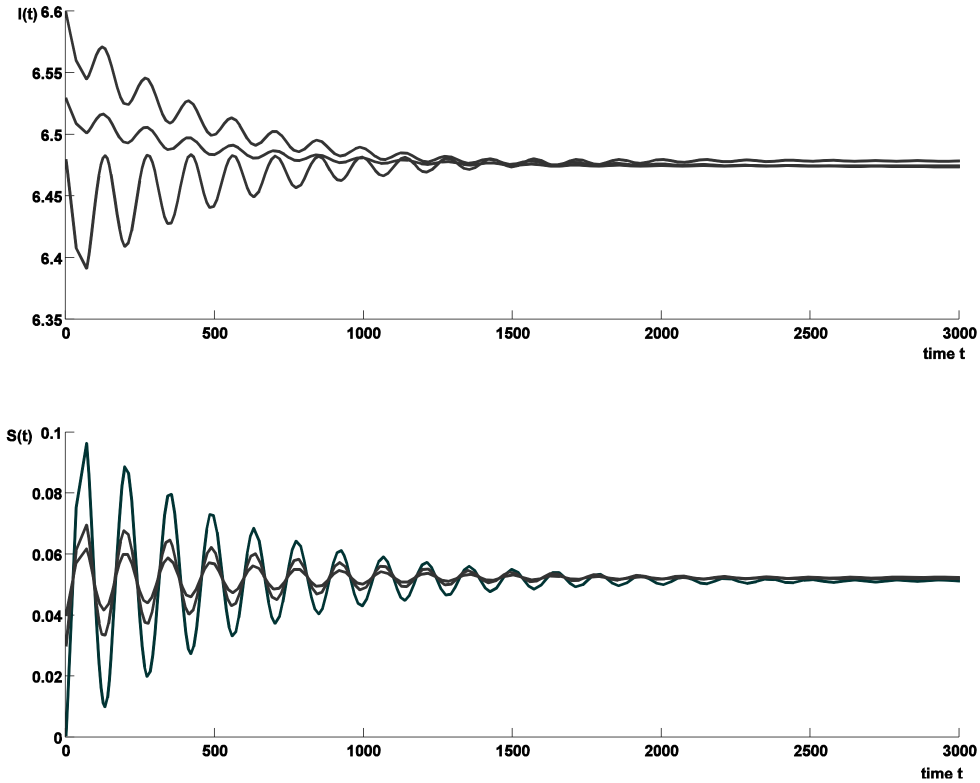
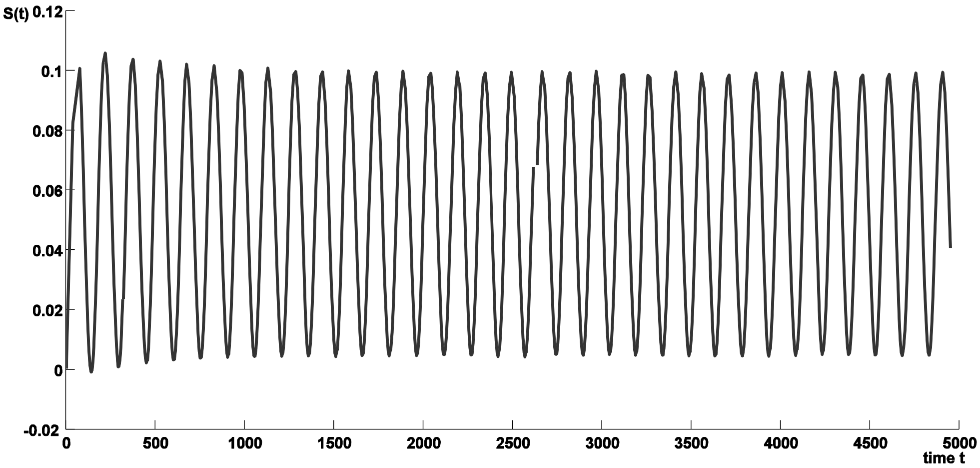


Figure 1. $\tau = 35 < \tau_0$, $R_0 > 1$. The solutions of (1.4) tends to the positive equilibrium E^*.



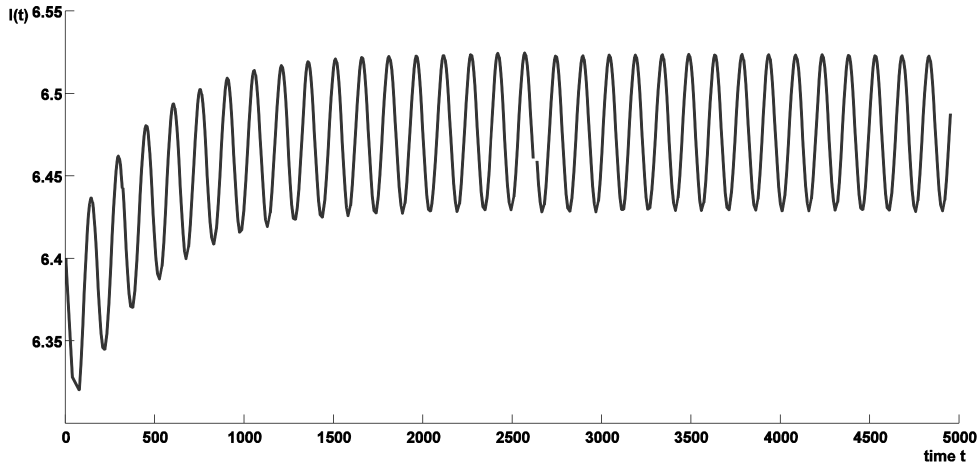


Figure 2. $\tau = 38.22 > \tau_0$, $R_0 > 1$. Hopf bifurcation occurs at the positive equilibrium E^*.

6. Conclusions

By analyzing the transendental characteristic equation, we derived stability conditions for the positive equilibrium in terms of the time delay τ.

When $\tau = 0$, the positive equilibrium E^* has the following properties:

- If $p_1 + q_1 > 0,\ p_2 + q_2 > 0$ and $\Delta \geq 0$, E^* is a stable note;
- If $p_1 + q_1 > 0,\ p_2 + q_2 > 0$ and $\Delta < 0$, E^* is a stable focus;
- If $p_2 + q_2 < 0$, E^* is a saddle;
- If $p_1 + q_1 < 0,\ p_2 + q_2 > 0$ and $\Delta \geq 0$, E^* is an unstable note;
- If $p_1 + q_1 < 0$ and $\Delta < 0$, E^* is an unstable focus.
- If the positive equilibrium E^* is stable when $\tau = 0$, then, for $\tau > 0$, it is always stable if one of the following conditions holds:
- $(2p_2 + q_1^2 - p_1^2)^2 - 4(p_2^2 - q_2^2) < 0; 2p_2 + q_1^2 - p_1^2 < 0,$
- $p_2^2 - q_2^2 \leq 0$ and $(2p_2 + q_1^2 - p_1^2)^2 - 4(p_2^2 - q_2^2) \geq 0.$

By virtue of *Nyquist criterion*, the length of the delay τ_+, which preserves the stability of the positive equilibrium is estimated, *i.e.* the positive equilibrium is asymptotically stable for $\tau < \tau_+$.

Using the method developed in [28]-[29], the stability, the direction and the periodic of bifurcating period solutions are discussed at $\tau = \tau_j$, $j = 0,1,2,\cdots$. The existence of the period orbits suggest that the latent period brings out the periodic oscillations of both the numbers of the susceptible plants and the infected ones, which help us to understand why some diseases outbreak periodically in infection period.

Employing the parameter values presented in [5] and the references, we carried out some simulations to support our qualitative results. Choosing $\phi = 0.07$, $p = 0.0064$, we obtain $\tau_0 = 38.21688903$, $k_2 = 0.5430753270$, and $b_2 = -0.0002645313913$, which implies that a stable supercritical Hopf bifurcation occurs at the positive equilibrium E^* when $\tau > 38.22$, and E^* is stable when $\tau < 38.22$. Seen from Fig. 1 that the number of the susceptible plants and the infected ones oscillation near E^* and then tend to it as *t* increasing when $\tau = 35$. Fig. 2 shows that the number of the susceptible plants and the infected ones periodically oscillation around E^* when $\tau = 38.22$, and Hopf bifurcation occurs at E^*.

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