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A Brief Introduction to Special and General Relativity

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Abstract

This report would briefly introduce the theories of Special and General Relativity to audiences, and hopefully motivates audiences to continue further studies in this subject.

Keywords

Special and General Relativity, theory.

1. Introduction

Einstein's theory of Relativity has always been the fundament of science research and proven by scientists to be correct for countless times ever since its publicity almost a century ago. Although it is now a commonly used tool by scientists to explain and predict many phenomena in the universe, many of its ideas are still foreign to the general public and science lovers at a high school level. However, before this report introduces this fundamental theory in an easily understandable way, we should first know that why the theory of Relativity even exists.

Firstly, lets define the term "inertial" as something that is stationary or traveling at a constant velocity, and an "observer" as an object carrying a set of measuring tools that could be stationary or in motion. Imagine an inertial spacecraft traveling at a constant speed, and two masses are thrown from the spacecraft, one in the direction which the spacecraft is traveling at, and the other in the opposite direction. In this scenario, seen from a stationary observer outside the spacecraft, the two masses should have different speeds. This was described by Isaac Newton very clearly and should be quite intuitive. Now consider that two beams of light are emitted. According to the previous example of the two masses, the two beams of light should have different speeds as well. However, according to experiments first done by Albert Michelson and Edward Morley, the speed of light, c, is constant in all reference frames. Hence, it seems like we need a new theory that extends upon Newton's to describe light, or more generally, objects traveling at or near the speed of light. This theory is Relativity as we know it.

With that being said, this report would briefly introduce the theories of Special and General Relativity to audiences at a high school level, and hopefully motivates audiences to continue further studies in this subject.

2. Motion in Newtonian Terms

In this section we would discuss the movement of objects in Newtonian terms and how the Newtonian theory is incorrect under certain circumstances. That error eventually leads to the concept of Relativity and will be discussed in the next section.

2.1. Inertial Frames

It is very common for physics problems we see in textbooks to assume an object is moving in a straight line at a constant speed. But what exactly is a "straight line" and "constant speed"? Two important concepts have to be introduced to answer this question. A reference frame, which is essentially a set of coordinates, and an event, which is something that happens in a definite time and definite place. The straight line, as defined in Newton's laws, is the shortest path between two points in the frame, represented by the following equation in Cartesian coordinates:

$$dS^2 = dx^2 + dy^2 + dz^2 \tag{1.1}$$

in which dS^2 is the square of the change in distance, and x, y, and z are Cartesian coordinates measured by an observer traveling on the straight line. Any frame that experiences no external force, and thus no acceleration, is called an inertial frame, whereas an accelerating frame is called a non-inertial frame.

A clock could be used to time the motion of the object in an inertial frame, and constant speed is defined as the second derivative of distance over time equal zero, namely

$$\frac{d^2x}{dt^2} = 0 \tag{1.2}$$

in the x direction. The same equation could be used for motions in other directions.

Two reference frames could also move relative to each other, and the coordinates of a point could be represented separately on these two frames. For instance, a second observer moves in the x direction at a speed of v, so its x coordinates x' relate to the first observer by the relation below:

$$x' = x - vt \tag{1.3a}$$

In this scenario, the frames are relatively stationary to each other in the y and z directions, so the y and z coordinates would be related to that of the first observer in the following way:

$$y' = y \tag{1.3b}$$

$$z' = z \tag{1.3c}$$

The set of equations (1.3) is also known as the Galilean Transformation, and it is compatible only when $v \ll c$. When v is not significantly smaller than c, the counterpart of the Galilean Transformation, the Lorentz Transformation, is used instead. This will be discussed further in section 2.2.

2.2. Addition of Velocities

Before answering the question why a larger speed changes the relationship between two inertial frames, let's talk more about motion in Newtonian terms and why it is not compatible for objects traveling at high speeds.

Suppose an object is moving at velocity V in an inertial frame where its speed in the x direction is represented as V^x. Thus, from the perspective of a reference frame that is moving at speed v in the x direction, the speed of the particle in the x direction is

$$V^{x'} = V^x - v \tag{1.4}$$

This is known as the Newtonian addition of velocities rule, as described in the introduction of this report.

There exists a hypothetical frame called the Ether in which photons travel at the speed of light, c. Given the Newtonian addition of velocities rule, the speed of photons must not be identical from the perspective of any inertial frame that is not relatively stationary to the Ether. Hence in this theory, the speed of photons could be greater than or less than c depending on the direction of motion of the inertial frame. However, Albert Michelson and Edward Morley published the results of an experiment in 1887 showing that the speed of photons were identical and constant in all directions: all traveling at c. This proved that the Newtonian theory to be not applicable for photons and that the Ether did not exist, and a new theory was needed.

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3. Special Relativity

3.1. Relativistic Spacetime

Firstly, several important concepts and useful tools for representing relativistic spacetime would be introduced. A spacetime diagram could be used to show the relative motion in this scenario. The horizontal axis shows the displacement of an object at a certain direction, which the vertical axis shows the time. Time here is represented by ct but not t in order to make the units of both axis identical for simplicity.

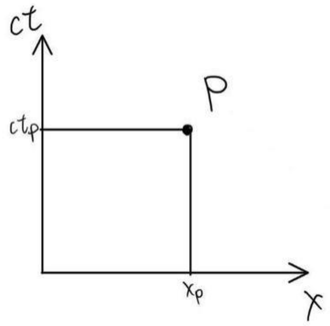


Fig 1. An example of a spacetime diagram. In this diagram, the event P happens at point x_p at the time t_p.

The worldline, which describes the motion of an object, is an important concept in spacetime diagrams. For instance, the worldline A in figure 2 represents an object which first accelerates and then travels at a constant speed, and the worldline B represents an object that remains still.

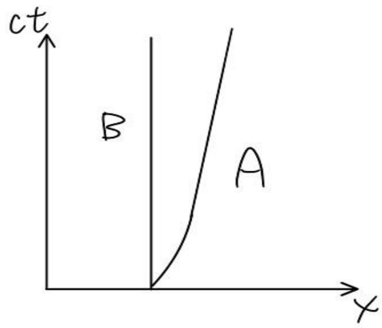


Fig 2. An example of two worldlines.

The reciprocal of the gradient of a worldline shows the velocity of the object. For a gradient larger than 1, a small change in ct is larger than a small change in x, meaning that the object is traveling slower than the speed of light. For a gradient smaller than 1 and larger than 0, a small change in ct is less than a small change in x, meaning that the object is traveling faster than the speed of light. A gradient equal 1 simply means the speed of light. Stationary objects have vertical worldlines.

Suppose flashes of light are emitted from an event O and travels in all directions. The worldlines of the light rays form a right angle above O, which together with the worldlines is called a light cone. All future events related to O could only be within the light cone as the speed of light is the fasted speed any object could travel at. Similarly, the past events related to O forms a right angle below O, as shown in Fig 4. These worldlines, together with the regions of spacetime they enclose, is called the light cone of event O.

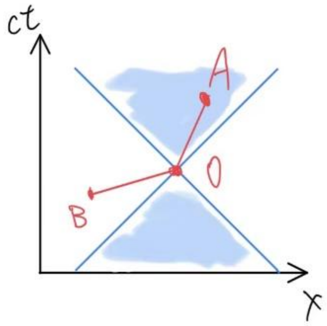
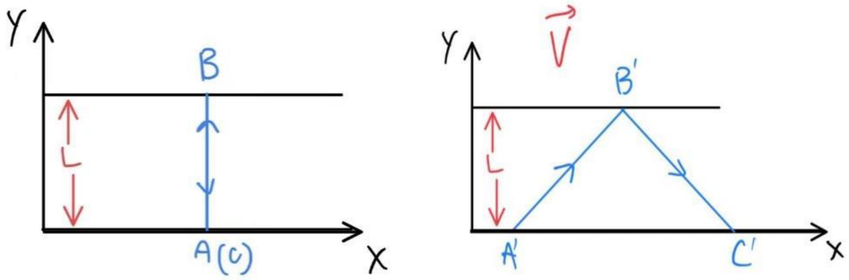


Fig 3. The light cone of an object O.

Point A is within the light cone, meaning that O could send signals to A. A could not send signals to O, however, because A is ahead of O in time. O and A are said to be timelike separated. On the other hand, B is out of reach of any signal from O, so O and B could not communicate with each other in any way. O and B are said to be spacelike separated. For an object right on the light cone of O, these two objects are said to be null separated.

Now there would be an example that helps understanding of spacetime under relativistic terms. Suppose there are two parallel mirrors separated by a distance L in the y direction. A beam of light is emitted from event A, and reflects at event B back to event C. In one inertial frame, the mirrors could be seen as stationary, while in another inertial frame, the mirrors are traveling in the x′ direction at a velocity of V, and in another inertial frame, the mirrors are stationary.



Figs 4. The mirror example as mentioned above. Left: the perspective of the first reference frame in which the mirrors are stationary, and the coordinates of the light beam are denoted as (t, x, y, z). Right: the perspective of the second inertial frame, and the coordinates of the light beam are denoted as (t', x', y', z').

In the first inertial frame in which the mirrors are stationary, distance traveled in all three directions equal 0, or

$$dx = dy = dz = 0 \tag{2.1}$$

Therefore, the total distance traveled could be written as

$$2L \tag{2.2}$$

and the total time needed is thus

$$\frac{2L}{c} \tag{2.3}$$

In the inertial frame which the mirrors are in motion, however, the coordinates are different. Since palpably the displacements $A'B' = B'C'$, they could be considered separately for simplicity. When traveling through $A'B'$, the displacement in the x' direction, the direction in which the frame is moving, is denoted as

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$$dx' = Vdt' \tag{2.4}$$

According to the Pythagoras theorem, therefore, the displacement $A'B'$ could be denoted as

$$\sqrt{L^2 + (\frac{dx'}{2})^2} \tag{2.5}$$

Hence, the total displacement could be written as

$$2\sqrt{L^2 + (\frac{dx'}{2})^2} \tag{2.6}$$

and thus the time taken could be given as

$$\frac{2}{c}\sqrt{L^2 + (\frac{dx'}{2})^2} \tag{2.7}$$

Since there is no displacement in the y' and z' directions in this frame as well,

$$dy' = 0 \tag{2.8}$$

$$dz' = 0 \tag{2.9}$$

Substituting (2.1), (2.3), (2.4), and (2.7), it is easy to derive that

$$-(cdt')^2 + (dx')^2 = -(cdt)^2 \tag{2.10}$$

Hence

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{2.11a}$$

$$(ds)^2 = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \tag{2.11b}$$

Here, $(ds)^2$ is the invariant since it could be both positive and negative, which means it is identical in both reference frames. This invariance is analogous to the distance in Euclidean geometry as defined in equation (1.1). The concept of invariance is one of the most important concepts in Special Relativity, and it could together with the concept of spacetime define the Minkowski spacetime, another very useful concept.

This also means that for $\Delta s^2 < 0$, the distance that an object travels in a certain period of time is less than the distance that light travels. Thus, it is said to be inside the light cone, which means spacelike separation. Similarly, $\Delta s^2 > 0$ means timelike separation, and $\Delta s^2 = 0$ means null separation.

Given (2.11a), it is palpable that distances represented in a spacetime diagram do not necessarily match distances in Euclidean geometry.

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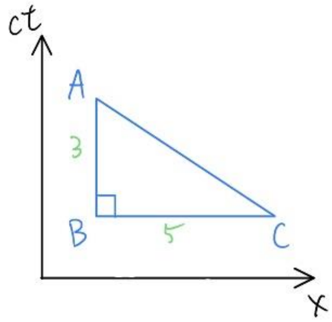

Fig 6. An example in which the spacetime in a spacetime diagram is different from the real spacetime.

In this spacetime diagram, the length of AB is 3 and the length of BC is 5. According to Pythagoras' Theorem on a Cartesian plane in Euclidean geometry, the length of $AC = \sqrt{AB^2 + BC^2} = \sqrt{34}$. However, since this is in a spacetime diagram, according to (10.2), the interval (represented as $A'C'$ to show the difference) could be given as $\sqrt{-AB^2 + BC^2} = 4$. This invariant interval $A'C'$ is both shorter than AC in a Cartesian plane and BC on the diagram.

3.2. Lorentz Transformation

Consider this question: how are coordinate systems in two inertial frames in relative motion related to each other? The idea of transforming between these two coordinates is finding an invariant and carry out the deduction of the coordinate transform based on that it. This leads to the idea of Lorentz Transformation, under which the speed of an object could be seen as an invariant property. If one wishes to know how are two frames related to each other in relativistic terms, the key is to first let the two frames agree on a common invariant interval. With that being said, the coordinates (t', x', y', z') in an inertial frame traveling at speed v in the x direction is related to the coordinates (t, x, y, z) in a stationary frame by the following equations:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.12a}$$

$$x' = \gamma(x - vt) \tag{2.12b}$$

$$y' = y \tag{2.12c}$$

$$z' = z \tag{2.12d}$$

in which γ is the Lorentz factor, namely

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.13}$$

The two frames in the scenario of the set of equations (2.12) are relatively stationary in the y and z directions. This set of equations could be adjusted for different speeds in the y and z directions as well. This is known as the Lorentz Transformation.

Note that for $v \ll c$, the differences caused by γ could be neglected since γ becomes small, so these equations above change to the Galilean Transformation (transformation in non-relativistic situations) instead.

In the next sections we would show the significance and importance of Lorentz Transformation in understanding Relativity and include examples of its application.

3.3. Constant Speed of Light

The above equation is one of the fundaments of Special Relativity. Another key feature that leads to Special Relativity is the inexistence of the Ether, as mentioned previously. Therefore, another equation modeling the addition of relativistic velocities is needed, and we could use the Lorentz Transformation to derive it.

Consider the case that Bob is in an inertial frame traveling at speed v_f in the x direction and he throws an object in a straight line at speed v in the same direction as the motion of the inertial frame. Alice is in a stationary frame. Using the Newtonian addition of velocities rule, the speed of the object is simply $v' = v_f + v$, but as v_f and v increases to relativistic speeds, we have to consider the Lorentz Transformation. With that being said, we could get that

$$v' = \frac{x'}{t'} = \frac{x_f + vt}{t_f + \frac{vx_f}{c^2}} \tag{2.14a}$$

by substituting equations (2.12a) and (2.12b), where x_f is the distance traveled by Bob's reference frame and t_f is the time the frame used. The sign change indicates that v_f and v are in the same direction. By dividing both the numerator and the denominator by t_f, we could get

$$v' = \frac{\frac{x_f}{t_f} + v}{1 + \frac{v \times \frac{x_f}{t_f}}{c^2}} \tag{2.14b}$$

replacing $\frac{x_f}{t_f}$ with v_f, we could finally have

$$v' = \frac{v_f + v}{1 + \frac{v_f \times v}{c^2}} \tag{2.14c}$$

This formula works for $v < c$ as well, and the reader could easily find that even if v_f and v are both less than c, the result is still different from that calculated with the Newtonian addition of velocities rule. When $v = c$, $v' = c$ regardless of the value of v_f. This is known as the Einsteinian addition of velocities rule.

Having a constant speed of light in all reference frames could lead to problems. Assume that there is a rocket traveling at a constant speed v in the x direction relative to a nearby planet. Alice is standing at the front of the rocket while Bob is standing at the back of the rocket. Using the rocket as reference frame, when Alice and Bob both shoot a laser at a detector O that is right in the middle of the rocket, O would receive the two lasers simultaneously. Using the nearby planet as reference frame, however, the rocket is relatively moving, so the laser from Bob has to travel a longer distance than the laser from Alice, since it could be regarded as that O is catching up with Alice and moving away from Bob at speed v. That means if O is to receive the two lasers simultaneously, Bob would have to emit her laser earlier than Alice does, since both lasers travel at speed c. Hence, two simultaneous events in one inertial frame may not necessarily be so in another. This is known as the relativity of simultaneity, a very important concept in Special Relativity.

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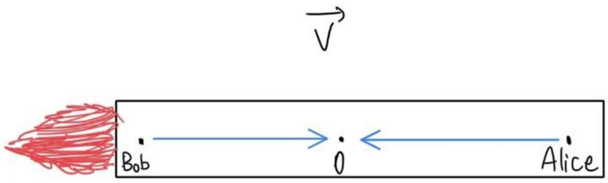

Fig 5. Graph showing the rocket example. The distances the lasers from Alice and Bob have to travel in order to reach O is the same in the rocket frame, but not in the planet frame since there is relative motion.

3.4. Time Dilation and Length Contraction

A clock measures time, and thus timelike distances, corresponding to a ruler, which measures spacelike distances. Therefore, a clock could also be seen as to measure the length of a segment of a worldline, which the following formula could be used:

$$d\tau^2 \equiv -\frac{ds^2}{c^2} \tag{2.15}$$

in which dτ is the length of the segment of the worldline, also known as proper time. This equation applies to two near events.

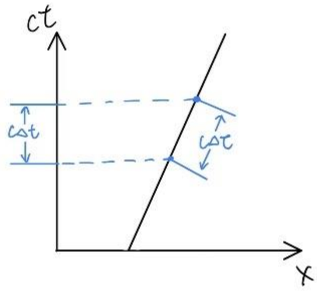

Fig 7. Graph comparing proper time and coordinate time. As seen in the graph, the change in coordinate time, Δt, is less than the change in proper time. This is called "time dilation" and will be explained further below.

The proof and the mathematical formula of time dilation would be derived below. Substituting (2.11a) into (2.13), it could be derived that

$$d\tau^2 = -\frac{-(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2}{c^2} \tag{2.16}$$

$$d\tau^2 = -\left\{-(dt)^2 + \frac{1}{c^2}[(dx)^2 + (dy)^2 + (dz)^2]\right\} \tag{2.17}$$

$$d\tau^2 = (dt)^2\left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\} \tag{2.18}$$

and in terms of space only, we could substitute $V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ into (2.18) to get

$$d\tau^2 = (dt)^2\left(1 - \frac{V^2}{c^2}\right) \tag{2.19}$$

and therefore

$$d\tau = dt\sqrt{1 - \frac{V^2}{c^2}} \tag{2.20}$$

which could be rearranged as the following:

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$$dt = \frac{d\tau}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (2.21)$$

For simplicity, we could substitute (2.13) into (2.21) to get the following:

$$dt = \gamma d\tau \quad (2.20)$$

Since for any $V < c, \gamma > 1$ and thus $dt > d\tau$, meaning that the time along a timelike worldline happens to be less than the time in a stationary inertial frame. This which leads to the name “time dilation”. Note that this formula malfunctions on a spacelike worldline as the term under the square root $1 - \frac{V^2}{C^2}$ would be less than zero.

The Lorentz factor could be applied not only to time, but to length as well. The proper length (length of the object along a timelike worldline), L', is smaller by a factor of γ than the length of an identical object measured in a stationary inertial frame, L. This means that an object traveling at relativistic speeds seems to be smaller than another identical object which is stationary. Therefore, this is known as length contraction.

Due to time dilation and length contraction, we could not represent an inertial frame and a stationary frame in a spacetime diagram using the same set of axes anymore. We would have to introduce a new set of axes to show the two frames on the same spacetime diagram. Note that the blue axes in Fig 8 is the result of a Lorentz Transformation of the two black axes.

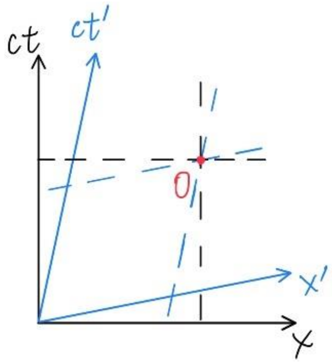

Fig 8. A spacetime diagram with two sets of axes. In this diagram, ct and x represents coordinates in the stationary frame, and ct' and x' represents coordinates in the moving inertial frame. Note that here the displacement along the y and z directions are assumed to be zero.

From equations (2.12a) and (2.12b), it is straightforward to see that x' and t' are mutually independent. Therefore, the x' axis in Fig 8 is the axis on which $ct' = 0$, and vice versa. This determines that both the x' and ct' axes should be straight lines.

As seen from Fig 8, the proper time and proper length of the event O is different from its time and length measured in a stationary inertial frame. This is how coordinates of an event in two inertial frames could be represented in one spacetime diagram only.

3.5. Interesting Problems that Arise

Time dilation could lead to interesting problems. For instance, suppose there are two twins, Alice and Bob. Alice stays on Earth while Bob sets off on a return journey to Alpha Centauri (a star about 4.4 lightyears away from Earth) on a spaceship that could instantly accelerate to 0.8c and decelerate back to 0. Due to time dilation, using Earth as the rest frame, when Bob arrives back on Earth, he would experience a shorter time than Alice, who stays on Earth, so Bob would be younger than Alice. Alternatively, we could look at this from Bob’s rest frame, and Alice was traveling at 0.8c away from Bob, so in this case Alice would be the younger sibling. Either argument seems to be logical, but Alice could not be younger and older than Bob at the same time. This is the famous twin paradox.

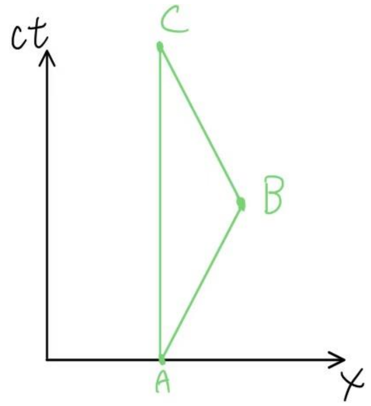

Fig 9. A spacetime diagram showing the twin paradox seen from Alice's perspective. AC is Alice's worldline, and AB and BC is Bob's worldline. In the diagram, event A is Bob's departure from Earth, event B is when Bob reaches Alpha Centauri, and event C is when Bob arrives back on Earth.

One could easily notice that at B, Bob undergoes an acceleration that changes the direction of his velocity. Due to this acceleration, Bob's and Alice's worldlines are not symmetrical anymore, so one could not simply reverse the two. As a result, the twin paradox is not really a paradox, and Bob would be younger than Alice when he returns. The age difference at event C when Bob returns to Earth is trivial and left as an exercise for the reader.

Another interesting example would be a runner running at near the speed of light carrying a 20m long pole in the direction of motion. In front of him is a 10m long barn, with both doors on the path of the runner and wide open. Assume the runner is running at a speed that from the barn's perspective, the pole is contracted to only 10m long, so it the two doors could close simultaneously and the pole would fit perfectly fine. However, from the runner's perspective, the barn would be only 5m long, and enclosing the pole inside the barn would seem impossible. However, this is simply another misconception and could be solved using a spacetime diagram as well.

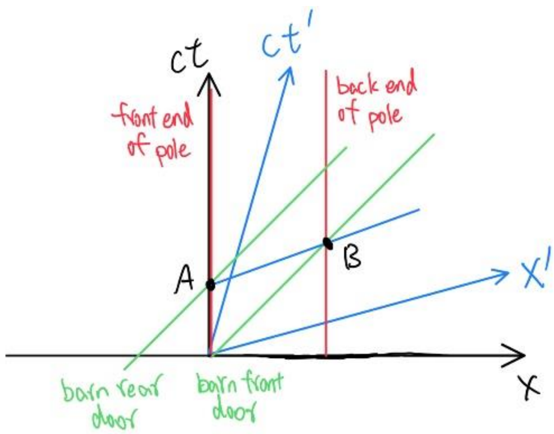

Fig 9. Spacetime diagram of the pole and barn example, seen from the pole and the runner's perspective. Hence, seen from this perspective, the runner and the pole is in a rest frame, and the barn is moving towards them.

As palpably seen, from the barn's perspective, the time when the front door and rear door closes (events A and B) are the same, and these two events are simultaneous. However, from the runner's perspective, these two events are not simultaneous, and event A happens before event B. Therefore, although to the barn the pole has already been enclosed in it, to the runner parts of the pole are still outside the barn, and the rear door is not closed.

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4. The Equivalence Principle

Now that we have briefly introduced Special Relativity, it's time to move onto other parts of Relativity, namely General Relativity and its applications. However, before we do that, one essential concept, the equivalence principle, has to be discussed first. The equivalence principle is one of the most key contributors to General Relativity, which would be covered in the next section. This section of the report would briefly introduce the equivalence principle and its applications.

4.1. Two Different Types of Mass

People generally regard the mass of an object as a single feature. However, there are actually two types of mass, gravitational and inertial, and they have different meanings and significances.

Gravitational mass measures how strongly two objects attract, given by

$$F = \frac{GMm}{R^2} \tag{3.1}$$

Inertial mass measures the inertial of an object, given by

$$F = ma \tag{3.2}$$

Given equations (3.1) and (3.2), we could easily cancel out the two ms for the same object measured, namely

$$a = \frac{GM}{R^2} \tag{3.3}$$

Hence, the acceleration, or the fall rate in a gravitational field, of an object is independent of the mass of the object, and only depend on the strength of the gravitational field. This was first proved to be true by Galileo a few centuries ago by rolling balls of the same size but different masses off an inclined plane and discovered that the masses of the balls had no direct relationship with their accelerations.

In reality, there is no way one type of mass of an object could be measured individually without the other type. However, many experiments have proven that the gravitational mass and inertial mass of an object are in fact equal. This is known as the equivalence principle.

In order to have a clearer understanding of the significance of the equivalence principle, consider Alice sitting in a rocket and feeling a force pulling her backwards. According to the equivalence principle, any mechanical experiment (e.g., projectile experiment, lever experiment, etc.) she conducts on the rocket would not be able to let her determine whether the rocket is in constant acceleration, or she is sitting on a rocket that is itself not moving but rather in a gravitational field, e.g., on Earth. In other words, the effects of a constant gravitational field and a constant acceleration of the same magnitude on the same object are identical. This is a very important step that leads to General Relativity in Section 4.

4.2. Light Rays in a Gravitational Field

In the previous section we discussed that any kind of mechanics experiment could not distinguish between a gravitational field and a constant acceleration. However, what if the experiment was not about mechanics? Maybe in this case, we could go beyond known physics and assume that the equivalence principle is also valid for all physics, and see what that teaches us.

Suppose a rocket with two openings at different heights, one on each side, is flying in the z direction at a constant velocity, as shown in the graph below. A beam of light enters the rocket through the opening on the left. However, during the time it takes the beam of light to travel through the rocket, the rocket itself has gone up a bit from its original position. Therefore, instead of hitting the wall of the rocket at the same level as it entered, the beam of light exits the rocket through the second opening on the right, slightly below the first opening.

Now the scenario changes to an accelerating rocket, so the reference frame becomes non-inertial. The equivalence principle, which was assumed earlier to be valid for light too, has to be taken into account. Hence, a generalized version of the equivalence principle could be used to predict the behavior of light in constant acceleration, which we could then use to deduce how light might behave in a gravitational field.

In this case, the entry point of the beam of light would also be higher than the exit point, but seen from the rocket frame, the light would no longer travel in a straight pass, but rather a curved one.

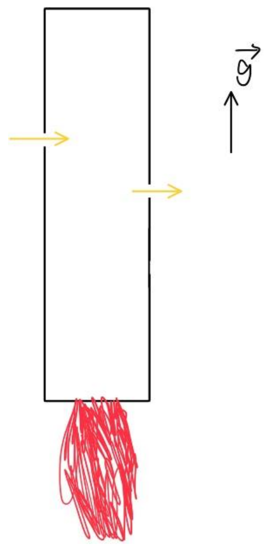

Fig 10. The graph of the example above. This graph is not to scale, and has been exaggerated to make the effects more significant. Note that the measurements in the example are taken from the rocket frame, which is inertial.

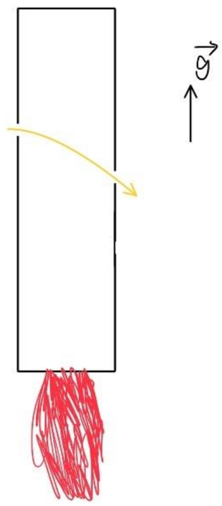

Fig 11. The curvature of the beam of light observed by an observer on an accelerating rocket. Note that this effect could also be observed by rocket at rest in a gravitational field, according to the equivalence principle.

From the example above, we could learn that light bends under uniform accelerations. Therefore, based on the previous assumption of the equivalence principle, it is then safe to conclude that gravity bends light, which turn out to be a key feature of General Relativity.

4.3. Clocks in Gravitational Fields

Now let's consider the scenario where Alice stands at the top of a rocket traveling at a constant acceleration, and Bob stands at the bottom of the rocket. Alice sends a signal towards Bob, and the time interval between the sending and the receiving of the signal t is measured using a clock in an inertial frame outside of the rocket.

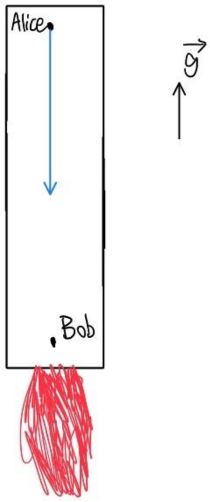

Fig 12. Alice standing at the top of a constantly accelerating rocket sending a signal to Bob, who is standing at the bottom of the rocket.

For a rocket traveling at a constant acceleration, the time interval between each signal received by Bob, τ' is greater than the time interval between each signal sent by Alice, τ. This is because the light beam is received at shorter intervals than they are emitted on the accelerating rocket, since Bob catches up with the signal slowly as the it travels towards Bob. This effect is not seen on the non-accelerating rocket. Therefore, together with the equivalence principle, it is safe to conclude that signals are received at a faster rate than they are emitted in a gravitational field.

The magnitude of this effect could be calculated in the following way. Bob's position could be denoted as

$$z_B = \frac{1}{2}gt^2 \tag{3.4}$$

Hence, Alice's position could be denoted as

$$z_A = h + \frac{1}{2}gt^2 \tag{3.5}$$

where h is the length of the rocket. Suppose the signal is emitted at $t = 0$ and received at t_1. Therefore, the distance traveled by the first signal is

$$z_A(0) - z_B(t_1) = ct_1 \tag{3.6}$$

Using $\Delta\tau_A$ as the time the second signal is emitted and $t_1 + \Delta\tau_B$ as the time it's received where $\Delta\tau_B$ is a time interval, assuming $\Delta\tau_A$ and $\Delta\tau_B$ are small that we only need to consider linear terms, the distance traveled by the second signal could be denoted as

$$z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A) \tag{3.7}$$

Equations (3.5) and (3.6) could be expanded to get

$$h-\frac{1}{2}\mathrm{g}{\mathrm{t}_1}^2=\mathrm{ct}_1 \tag{3.8a}$$

$$h+\frac{1}{2}\mathrm{g}\Delta{\tau_\mathrm{A}}^2-\frac{1}{2}\mathrm{g}{\mathrm{t}_1}^2-\mathrm{g}\mathrm{t}_1\Delta\tau_\mathrm{B}-\frac{1}{2}\mathrm{g}\Delta{\tau_\mathrm{B}}^2=\mathrm{c}(\mathrm{t}_1+\Delta\tau_\mathrm{B}-\Delta\tau_\mathrm{A}) \tag{3.8b}$$

respectively. Since $\Delta\tau_\mathrm{A}$ and $\Delta\tau_\mathrm{B}$ are small, (3.7b) could be simplified as

$$h-\frac{1}{2}\mathrm{g}{\mathrm{t}_1}^2-\mathrm{g}\mathrm{t}_1\Delta\tau_\mathrm{B}=\mathrm{c}(\mathrm{t}_1+\Delta\tau_\mathrm{B}-\Delta\tau_\mathrm{A}) \tag{3.8c}$$

Using (3.8a) and (3.8c), the following result could eventually be derived

$$\Delta\tau_\mathrm{B}=\Delta\tau_\mathrm{A}\left(1-\frac{\mathrm{g}h}{\mathrm{c}^2}\right) \tag{3.9}$$

Now that equation (3.9) has been derived, the equivalence principle could be applied to real-world scenarios.

Since $\mathrm{g}h$ is simply the difference in gravitational potential, the it satisfies the following relationship:

$$\Phi_\mathrm{A}-\Phi_\mathrm{B}=\mathrm{g}h \tag{3.10}$$

in which Φ is the gravitational potential. Therefore, we have

$$\Delta\tau_\mathrm{B}=\Delta\tau_\mathrm{A}\left(1-\frac{\Phi_\mathrm{A}-\Phi_\mathrm{B}}{\mathrm{c}^2}\right) \tag{3.11}$$

Moreover, the rates of emission, ω_A and ω_B, could simply be represented by $1/\Delta\tau_\mathrm{A}$ and $1/\Delta\tau_\mathrm{B}$ respectively. Therefore, the following formula could be derived:

$$\text{emitting rate}=\left(1-\frac{\Phi_\mathrm{A}-\Phi_\mathrm{B}}{\mathrm{c}^2}\right)\times\text{receiving rate} \tag{3.12}$$

From (3.12) it is easy to note that the receiving rate would be higher than the emitting rate. Therefore, Bob would observe a blueshift effect on the signals he receives. On the other hand, if in this case Bob sends a signal to Alice instead, then Alice would observe a redshift effect. The signals used in this equation is only a tool to assist comprehension, and not necessary to observe this effect. This effect also matches the predictions of the equivalence principle.

4.4. Examples of Application of the Equivalence Principle

Now that the fundamentals of the equivalence principle have been introduced, this section would discuss the equivalence principle in a scenario that is closer to real life.

Although very minor, the effects of the equivalence principle could be measured in our daily lives. Suppose Alice and Bob works at the same company. Alice's office is on the ground floor, but Bob's office is in higher floors about 100 meters above the ground. Therefore, substituting 100 meters into (3.9) and (3.10), one could know that measured from the ground, Bob's heartbeat would be faster than Alice's by a magnitude of

$$1+\frac{9.81\mathrm{ms}^{-2}\times 100\mathrm{m}}{(3\times 10^{8}\mathrm{ms}^{-1})^{2}} \tag{3.13}$$

That is approximately 1.09×10^{-14}. In other words, Bob would age faster than Alice by a magnitude of 1.09×10^{-14} simply because he is sitting at a level higher than her.

This effect could also be observed in light. Suppose a star emits light at a frequency of ω. Since the gravitational potential of the star is $\Phi=-\frac{GM}{R}$, the frequency of the light measured from a distant observer is then given by

$$\omega'=\left(1-\frac{GM}{Rc^{2}}\right)\omega \tag{3.14}$$

This effect is similar to the redshift caused by the Doppler Effect although having a completely different mechanism, and it is known as the gravitational redshift since it is caused by gravity.

The Global Positioning System (GPS) also applies the equivalence principle. GPS satellites travel at high velocities around the Earth at different altitudes. Therefore, the signals they send to each other experience differences in time and must be tuned to function correctly.

Firstly, how the GPS functions and its sources of error must be introduced. In one dimensional spacetime, two satellites are needed to know the exact location of a point, as shown in the graph below.

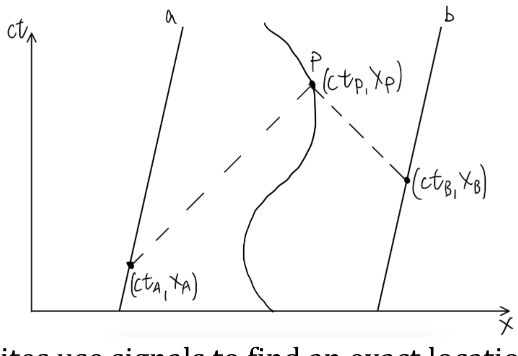

Fig 13. The way two satellites use signals to find an exact location along a worldline in one dimensional spacetime. The straight lines a and b are the worldlines of two satellites.

Each satellite sends a signal that includes the coordinates of its place of emission. The coordinates of the point P where the two signals are received simultaneously are given by the following equations:

$$ct_p=\frac{1}{2}[c(t_A+t_B)+(x_B-x_A)] \tag{3.15a}$$

$$x_p=\frac{1}{2}[c(t_B-t_A)+(x_B+x_A)] \tag{3.15b}$$

There are two factors influencing the difference in time between different satellites in the GPS: time dilation (Special Relativity) and the Earth's gravitational effects (General Relativity).

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Given (2.19), it is easy to know that the fractional correction $\eta_{\text{time dilation}}$ needed to compensate the effects of time dilation is

$$\eta_{\text{time dilation}} \approx \frac{1}{2}\left(\frac{V}{c}\right)^2 \quad (3.16)$$

in which V is the velocity of a satellite.

Given (3.11), it is also easy to know that the fractional correction $\eta_{\text{gravitational potential}}$ needed to compensate the effects of gravitational potential is

$$\eta_{\text{gravitational potential}} \approx \frac{GM}{Rc^2} \quad (3.17)$$

in which M and R is the mass and radius of the Earth respectively. In order to compare equations (3.16) and (3.17), we could use the following formula from Newtonian gravity:

$$\frac{V^2}{R} = \frac{GM}{R^2} \quad (3.18)$$

which shows the centripetal acceleration of the satellite. Note that from (3.17) and (3.18), the effects of gravitational potential are twice as significant as the effects of time dilation. Therefore, in order to have a properly functioning GPS system, both effects have to be considered.

General Relativity

Now the report has covered much about Special Relativity and the equivalence principle, and the topic would shift to General Relativity. This section would briefly introduce General Relativity and its relationship with Newtonian gravity. General Relativity is actually simply a name for Einstein's counterpart of Newton's theory of gravity, or Einstein's theory of gravity. From Einstein's theory, we could learn that gravity was in fact not a force, and the motion of objects once believed to be caused by that force are in reality caused by the curvature of spacetime.

4.4.1 Influence of Gravitational Potential on Proper Time

Equation (2.11a) is already a function that defines spacetime quite well. However, it still does not cover all aspects of Newtonian gravity and General Relativity, since it only applies to situations where gravity is negligible. Hence, a more generalized equation is necessary. The idea of General Relativity could be added onto (2.11a) to get

$$ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)(cdt)^2 + \left(1-\frac{2\Phi}{c^2}\right)[(dx)^2 + (dy)^2 + (dz)^2] \quad (4.1)$$

Note that one limitation of (4.1) is that it only works for weak gravitational fields, but that essentially covers a majority of gravitational fields present in the universe.

This could be proven as the following. Suppose signals are emitted at x_A and received at x_B respectively. The spacetime diagram of these two events could be pictured as Fig 14 as shown below.

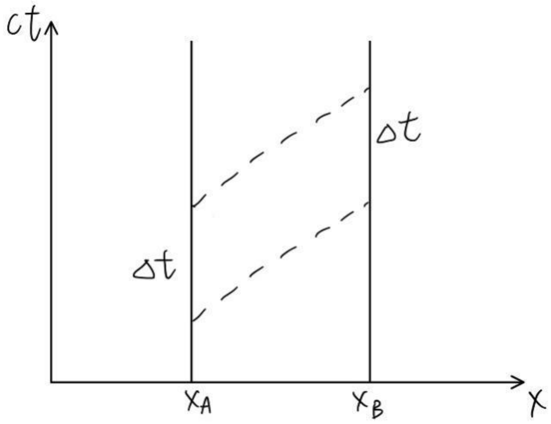

Fig 14. Spacetime diagram of two emissions, showing the emitter, the receiver, and two signals sent. Since in this case the emitter and the receiver at x_A and x_B are both constant observers, there should be two constant straight lines representing their worldlines. The worldlines of the signals would not be straight lines with gradient equal one as in a flat spacetime, but they should have identical shapes since it is not dependent on time.

Here, Δt represents the same coordinate time interval on the two worldlines. However, the proper time interval between the two points on the worldlines are not the same due to difference in gravitational potential. Since the event of the signals were emitted happen at only a point in time, the displacement along all three directions in the space dimension is zero, as described by (2.1), or $dx = dy = dz = 0$. Equation (4.1) becomes

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 \tag{4.2}$$

after substituting the displacement along the space dimension. Further substituting (2.13) into (4.2), it is not hard to find that

$$\Delta\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right)\Delta t^2 \tag{4.3}$$

In order to calculate the respective gaps in time for the emitter and the receiver, we could take the square root form both sides of equation (4.3). Moreover, Since the value of $\frac{2\Phi}{c^2}$ is small, we could apply it to the relation $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$ to it for a more simplified form:

$$\Delta\tau_A = \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t \tag{4.4a}$$

$$\Delta\tau_B = \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t \tag{4.4b}$$

Hence, the difference between the proper time intervals of the two emissions is given by eliminating Δt in both equations (4.4a) and (4.4b) to get

$$\Delta\tau_B = \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A \tag{4.5}$$

This agrees with the effects predicted by the equivalence principle in section 3. Therefore, we could conclude that General Relativity agrees with the equivalence principle.

The definition of proper time itself may vary when considering the effects of gravitational potential. More specifically, the proper time between events A and B could be denoted as

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-\frac{ds^2}{c^2}\right)^{1/2} \tag{4.6a}$$

after substituting (2.13). Expanding (4.6a),

$$\tau_{AB} = \int_A^B \left\{\left(1+\frac{2\Phi}{c^2}\right)(dt)^2 - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)[(dx)^2+(dy)^2+(dz)^2]\right\}^{1/2} \tag{4.6b}$$

Taking dt out,

$$\tau_{AB} = \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]\right\}^{1/2} \tag{4.6c}$$

Then a binomial expansion could be performed to equation (4.6c). The result could be therefore shown as the following:

$$\tau_{AB} \approx \int_A^B \left[1-\frac{1}{c^2}\left(\frac{1}{2}V^2-\Phi\right)\right] \tag{4.7}$$

Notice that (4.7) takes into account both the effects of time dilation and gravitational potential, just suitable for the GPS mentioned in the section 3.4. These effects were previously induced using the equivalence principle, but now we could see that they could also be derived using the idea of curved spacetime.

5. Conclusion

The theories of Special and General Relativity not only filled the gaps of Newtonian geometry, but also contributed to many other great theories in science and projects in engineering. Relativity is surely one of the greatest theories ever in the history of science and even the history of the human race.

The knowledge of Relativity extends way beyond the context of this report, and the purpose of this report is only to give readers a brief, easily-understandable introduction of Relativity. There are way more attracting phenomena that could be explained using Relativity: black holes, gravitational waves, and the Big Bang, to name a few. For instance, General Relativity proofed that mass warps spacetime, which then lead to the idea of an extreme contortion of spacetime caused by an almost infinite density that even light could not escape. This idea is commonly known as the black hole. And now, about a century after General Relativity was introduced, humans finally succeeded in taking pictures of two black holes, Sagittarius A* at the center of

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the Milky Way and M87* at the center of galaxy Messier 87, proving General Relativity to be valid once again.

General Relativity is also the fundament of various branches of physics. It predicts the expansion of the universe in cosmology, and it is they key to understanding reunification of forces such as string theory and inspired many ideas to sprout in particle physics.

Is Relativity enough to explain everything then? A short answer would be no, since our understanding of the universe is still limited. Still, it probably is not wrong given all the phenomena it does explain; it may be incomplete though, and improvements could probably be made to make it able to explain even more about the Universe.

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